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A bigraded version of the Weil algebra and of the Weil homomorphism for Donaldson invariants. Elementary algebra and cohomology behind the Baulieu–Singer approach to Witten's topological Yang–Mills quantum field theory

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Abstract

We describe a bigraded generalization of the Weil algebra, of its basis and of the characteristic homomorphism which besides ordinary characteristic classes also maps on cohomology classes leading to Donaldson invariants in the appropriate context. Furthermore these cohomology classes exhaust the image of the generalized characteristic homomorphisms.

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Introduction

In [2], a small differential algebra was introduced in connection with gauge fixing à la B.R.S. [3] in Witten's topological quantum field theory [13]. Furthermore, in the last part of [2, Section 5] this algebra was identified with a differential subalgebra of the algebra of differential forms on the product $P \times \mathcal{C}$ of a principal bundle P over a manifold M with the space $\mathcal{C}$ of principal connections of P. By a slight "abstraction" one can produce an algebra $\mathfrak{A}$ which is a free bigraded-commutative differential algebra which admits the above quoted

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differential algebra (of [2]) as homomorphic image. This algebra, which is described in Section 1 of the present paper, is obviously a bigraded version of the Weil algebra [12] (see also in [5,9]); it is a contractible algebra. Therefore the various cohomologies of $\mathfrak{A}$ as well as the ones of its images are trivial. It was claimed in [11] that the relevant cohomologies were the basic cohomologies for operations in the sense of Cartan [5,9]. Actually this is also implicit in [2, Section 5] since there, it is claimed that the relevant cohomology is the de Rham cohomology of $M \times \mathcal{C}/\mathcal{G}$ (and not the one of $P \times \mathcal{C}$) where $\mathcal{G}$ is the group of gauge transformations of P. It is the aim of this paper to show that, by using $\mathfrak{A}$ as generalization of the Weil algebra, one can generalize the Weil homomorphism in such a way that besides the usual characteristic classes, it makes contact with the Donaldson invariants [6]. That there is connection between the usual characteristic classes and the Donaldson invariant is natural in view of the apparencies of the Donaldson–Witten cocycles [13] which are bigraded expansions of the Chern–Weil cocycles. Our results imply that the classes of these bigraded expansions of Chern–Weil cocycles (i.e. the Donaldson–Witten cocycles) exhaust the image of the generalized characteristic homomorphism. In terms of fields this means that the (local) relevant basic BRS-cohomology of the bigraded differential algebra generated by the ghosts, the gauge potentials and their BRS and exterior differentials is exhausted by the above classes. It is expected that, as for the case of ordinary gauge theory [8], inclusion of arbitrary derivations of the fields does not modify essentially this result. It is worth noticing here that $\mathfrak{A}$ is a generalization of the Weil algebra based on the finite dimensional Lie algebra of the structure group of P and it is not the one based on the infinite dimensional Lie algebra of the group of gauge transformations of P [10]; in particular, $\mathfrak{A}$ is finitely generated.

The plan of the paper is the following. In Section 1 we describe the bigraded generalization of the Weil algebra and its basis and we compute its various basic cohomologies. In Section 2 we describe the corresponding appropriate generalization of the characteristic homomorphism. One can remark that the independence of the metric in [2] follows from the more general fact proved here that the characteristic homomorphism does not depend on the chosen connection of the space of connection over P (the structure group being there the gauge group). For a review on topological quantum field theory as well as for detailed references on the subject we refer to the Physics Report [4]. We apologize on the fact that we denote by α or more precisely by $\underline{\alpha}$ what is almost everywhere in the literature denoted by c, e.g. [2,4].

1. A universal model—generalization of the Weil algebra

1.1. Description of the model

Let $\mathfrak{g}$ be a finite dimensional Lie algebra and let (E_i) be a basis of $\mathfrak{g}$. Consider eight copies $\mathfrak{g}_A^*, \mathfrak{g}_F^*, \mathfrak{g}_\alpha^*, \mathfrak{g}_\varphi^*, \mathfrak{g}_\psi^*, \mathfrak{g}_\xi^*, \mathfrak{g}_B^*$ and $\mathfrak{g}_\beta^*$ of the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ with dual basis respectively denoted by $(A^i), (F^i), (\alpha^i), (\varphi^i), (\psi^i), (\xi^i), (B^i)$ and (β^i). Let $\mathfrak{A}(\mathfrak{g})$ (or simply $\mathfrak{A}$) be the free graded commutative algebra generated by the A^i and α^i in degree one, the F^i, φ^i, ψ^i and ξ^i in degree two and the B^i and β^i in degree three. On the space $\mathfrak{g} \otimes \mathfrak{A}$, there

is a natural bracket $[\bullet,\bullet]$ defined by $[X\otimes P, Y\otimes Q] = [X,Y]\otimes P\cdot Q$ for $X,Y\in\mathfrak{g}$ and $P,Q\in\mathfrak{A}$. For any linear mapping R of $\mathfrak{A}$ in itself one defines a linear mapping, again denoted by R, of $\mathfrak{g}\otimes\mathfrak{A}$ in itself by $R(X\otimes P) = X\otimes R(P)$ for $X\in\mathfrak{g}$ and $P\in\mathfrak{A}$. Let us introduce the following elements of $\mathfrak{g}\otimes\mathfrak{A}$ associated to the generators of $\mathfrak{A}$: $A = E_i\otimes A^i$, $F = E_i\otimes F^i$, $\alpha = E_i\otimes\alpha^i$, $\varphi = E_i\otimes\varphi^i$, $\psi = E_i\otimes\psi^i$, $\xi = E_i\otimes\xi^i$, $B = E_i\otimes B^i$ and $\beta = E_i\otimes\beta^i$. With the above notations, one has the following lemma.

Lemma 1.1. *There are unique antiderivations d and δ of $\mathfrak{A}$ satisfying $d^2 = 0$, $\delta^2 = 0$ and $d\delta + \delta d = 0$ such that $F = dA + \frac{1}{2}[A,A], \varphi = \delta\alpha + \frac{1}{2}[\alpha,\alpha], \psi = \delta A + d\alpha + [A,\alpha], \xi = \delta A, B = \delta F + [\alpha, F]$ and $\beta = d\varphi + [A,\varphi]$.*

Proof. By a change of generators, $\mathfrak{A}$ is the free graded commutative algebra generated by the A^i, α^i in degree one, $dA^i, \delta A^i, d\alpha^i, \delta\alpha^i$ in degree two and $\delta dA^i, d\delta\alpha^i$ in degree three. The lemma is thus obvious and $\mathfrak{A}$ is contractible for d and for δ. □

One defines an underlying bigraduation on $\mathfrak{A}$ by giving to the A^i the bidegree $(1,0)$, to the F^i the bidegree $(2,0)$, to the α^i the bidegree $(0,1)$, to the φ^i the bidegree $(0,2)$, to the ψ^i and the ξ^i the bidegree $(1,1)$, to the B^i the bidegree $(2,1)$ and to the β^i the bidegree $(1,2)$. $\mathfrak{A}$ is then a bigraded differential algebra with total differential $d+\delta$, d being the part of bidegree $(1,0)$ and δ the part of bidegree $(0,1)$. Since $\mathfrak{A}$ is contractible for each of these differentials its cohomologies are trivial ($H^{k,l}(\mathfrak{A},d) = 0$ if $k+l > 0, H^{0,0}(\mathfrak{A},d)\cong\mathbb{C}$, etc.). The interest of this bigraded differential algebra lies in the following universal property which is closely related to the triviality of its cohomology.

Lemma 1.2. *Let Ω be a bigraded commutative differential algebra with differential $d+\delta$, d of bidegree $(1,0)$ and δ of bidegree $(0,1)$. Let $\underline{A} = E_i\otimes\underline{A}^i$ be an element of $\mathfrak{g}\otimes\Omega^{1,0}$ and $\underline{\alpha} = E_i\otimes\underline{\alpha}^i$ be an element of $\mathfrak{g}\otimes\Omega^{0,1}$. Then there is a unique homomorphism of bigraded differential algebras $h:\mathfrak{A}\to\Omega$ such that $h(A^i) = \underline{A}^i$ and $h(\alpha^i) = \underline{\alpha}^i$, (in short $h(A) = \underline{A}, h(\alpha) = \underline{\alpha}$).*

Notice that $\mathfrak{A}$ contains two Weil algebras namely $W_d(\mathfrak{g}) = (\Lambda\mathfrak{g}^*_A\otimes S\mathfrak{g}^*_F, d)$ and $W_\delta(\mathfrak{g}) = (\Lambda\mathfrak{g}^*_\alpha\otimes S\mathfrak{g}^*_\varphi, \delta)$. The first one, $W_d(\mathfrak{g})$, will play in the sequel the role of the Weil algebra (see Section 2); its base (i.e. basic subalgebra) consists of the $\mathcal{I}(F)$ where $\mathcal{I}$ runs over all $ad^*(\mathfrak{g})$-invariant elements of $S\mathfrak{g}^*$ (i.e. invariant polynomials on $\mathfrak{g}$). Let $\mathfrak{B}(\mathfrak{g})$ (or simply $\mathfrak{B}$) be the set of elements of $\mathfrak{A}$ of the form $\mathcal{I}(F,\psi,\varphi,B,\beta)$ where $\mathcal{I}$ runs over all $ad^*(\mathfrak{g})$-invariant elements of $S\mathfrak{g}^*\otimes S\mathfrak{g}^*\otimes S\mathfrak{g}^*\otimes\Lambda\mathfrak{g}^*\otimes\Lambda\mathfrak{g}^*$ (invariant "polynomials" in F,ψ,φ,B and β). $\mathfrak{B}$ is a bigraded subalgebra of $\mathfrak{A}$ which is easily shown to be stable by d and by δ. For reasons which will become clear in Section 2, the bigraded differential algebra $\mathfrak{B}$ will be called *the base of* $\mathfrak{A}$.

The elements of $\mathfrak{B}$ are in fact basic for two operations of $\mathfrak{g}$ in $\mathfrak{A}$ which we now describe. One defines first an operation of $\mathfrak{g}$ in $\mathfrak{A}$, $X\mapsto i_X$, which extends the usual operation of $\mathfrak{g}$ in the Weil algebra $W_d(\mathfrak{g})$ by setting, for $X\in\mathfrak{g}$, $i_X(A) = X$ and i_X(other generators) $= 0$

which implies $i_X\delta+\delta i_X = 0$ and, with $L_X = i_X d+d i_X$, $L_X(Z) = [Z, X]$ for all generators $Z = A, F, \alpha, \varphi, \psi, \xi, B, \beta$. By interchanging the bidegrees, one defines similarly another operation of $\mathfrak{g}$ in $\mathfrak{A}$, $X \mapsto i'_X$, which extends the one of $W_\delta(\mathfrak{g})$. Namely one sets, for $X \in \mathfrak{g}$, $i'_X(\alpha) = X$, $i'_X(A) = i'_X(F) = i'_X(\varphi) = i'_X(\psi) = i'_X(B) = i'_X(\beta) = 0$ and $i'_X(d\alpha) = 0$ (notice that then $i'_X(\delta A) = i'_X(\xi) \neq 0$); this implies $i'_X d + d i'_X = 0$ and, with $L'_X = i'_X\delta + \delta i'_X$, $L'_X(Z) = [Z, X]$ for all generators $Z = A, F, \alpha, \varphi, \psi, \xi, B, \beta$. Let $\mathfrak{B}'$ be the common base for these two operations, i.e. remembering that $L_X \equiv L'_X$, $\mathfrak{B}'$ is defined by

$$\mathfrak{B}' = \{a \in \mathfrak{A} \mid i_X(a) = 0, i'_X(a) = 0 \text{ and } L_X(a) = 0, \forall X \in \mathfrak{g}\}.$$

One has $\mathfrak{B} \subset \mathfrak{B}'$ and $\mathfrak{B} = \mathfrak{B}'$ whenever the Lie algebra $\mathfrak{g}$ is semi-simple. However, when $\mathfrak{g}$ is not semi-simple, and in particular when $\mathfrak{g}$ contains an abelian factor, the inclusion $\mathfrak{B} \subset \mathfrak{B}'$ is a strict one.[3] It is clear that $\mathfrak{B}'$ is the natural base for $\mathfrak{A}$ considered as a bigraded version of the Weil algebra, nevertheless, what we call base in this paper is the bigraded differential subalgebra $\mathfrak{B}$. In fact, in Section 2, only the first operation will be mapped homomorphically on a similar operation of $\mathfrak{g}$. Nevertheless $\mathfrak{B}$ will be mapped into a basis of two operations.

In contrast to the case of $\mathfrak{A}$, the various cohomologies of $\mathfrak{B}$ are non-trivial and it is the aim of Sections 1.2 and 1.3 to compute these cohomologies.

1.2. The d and the δ cohomologies of $\mathfrak{B}(\mathfrak{g})$

Let d_1, d_2, δ_1 and δ_2 be the unique antiderivations of $\mathfrak{A}$ satisfying

$$d_1\psi = -B, \qquad d_1\varphi = \beta, \qquad d_1(\text{other generators}) = 0;$$

$$d_2 B = -[F, \psi], \qquad d_2\beta = [F, \varphi], \qquad d_2(\text{other generators}) = 0;$$

$$\delta_1 F = B, \qquad \delta_1\psi = -\beta, \qquad \delta_1(\text{other generators}) = 0;$$

$$\delta_2 B = [\varphi, F], \qquad \delta_2\beta = -[\varphi, \psi], \qquad \delta_2(\text{other generators}) = 0.$$

$\mathfrak{B}$ is stable by d_1, d_2, δ_1 and δ_2 and one has the following lemma.

Lemma 1.3. *The restriction of d to $\mathfrak{B}$ coincides with the restriction of $d_1 + d_2$ and the restriction of δ to $\mathfrak{B}$ coincides with the restriction of $\delta_1 + \delta_2$.*

Proof. One has $(d + \delta)(A + \alpha) + \frac{1}{2}[A + \alpha, A + \alpha] = F + \psi + \varphi$ which implies $(d + \delta)(F + \psi + \varphi) + [A + \alpha, F + \psi + \varphi] = 0$. By taking the homogeneous components and

[3] In the first version of this paper I claimed that $\mathfrak{B} = \mathfrak{B}'$ in general. The referee very rightly pointed out that this was a mistake and gave me the counterexample of an abelian Lie algebra $\mathfrak{g}$; I thank him very much.

by using the definitions, one obtains

$$\begin{cases} dF + [A, F] &= 0 \\ d\psi + [A, \psi] &= -B \\ d\varphi + [A, \varphi] &= \beta \\ dB + [A, B] &= -[F, \psi] \\ d\beta + [A, \beta] &= [F, \varphi] \end{cases} \quad \text{and} \quad \begin{cases} \delta F + [\alpha, F] &= B \\ \delta\psi + [\alpha, \psi] &= -\beta \\ \delta\varphi + [\alpha, \varphi] &= 0 \\ \delta B + [\alpha, B] &= [\varphi, F] \\ \delta\beta + [\alpha, \beta] &= -[\varphi, \psi]. \end{cases}$$

Thus the lemma would be obvious without the terms $[A, \cdot]$ and the terms $[\alpha, \cdot]$. However if $\mathcal{I}(F, \psi, \varphi, B, \beta) \in \mathfrak{B}$, it follows from the invariance of $\mathcal{I}$ that one can enter d in $\mathcal{I}$ by replacing it everywhere by $d + [A, \cdot]$ and that one can enter δ in $\mathcal{I}$ by replacing it everywhere by $\delta + [\alpha, \cdot]$. This implies the result. □

Lemma 1.4. *Let d_1' and δ_1' be the unique antiderivations of $\mathfrak{A}$ satisfying*

$$d_1' B = -\psi, \qquad d_1' \beta = \varphi, \qquad d_1'(\textit{other generators}) = 0$$

and

$$\delta_1' B = F, \qquad \delta_1' \beta = -\psi, \qquad \delta_1'(\textit{other generators}) = 0.$$

Then one has on $\mathfrak{B}$ $dd_1' + d_1' d$ = degree in $(\psi, \varphi, B, \beta)$ and $\delta\delta_1' + \delta_1'\delta$ = degree in (F, ψ, B, β).

Proof. One has on $\mathfrak{A}$

$$d_1 d_1' + d_1' d_1 = \text{degree in}(\psi, \varphi, B, \beta), \qquad d_2 d_1' + d_1' d_2 = 0,$$

$$\delta_1 \delta_1' + \delta_1' \delta_1 = \text{degree in}(F, \psi, B, \beta), \qquad \delta_2 \delta_1' + \delta_1' \delta_2 = 0.$$

Since both sides of these equations are derivations of $\mathfrak{A}$, it is sufficient to verify that they are true on the generators which is straightforward. The lemma follows then from Lemma 1.3. □

We are now ready to describe the d and δ cohomologies of $\mathfrak{B}$. Let $\mathcal{I}_S(\mathfrak{g})$ be the space of invariant polynomials on $\mathfrak{g}$ and let $\mathcal{I}_S^n(\mathfrak{g})$ be the subspace of homogeneous invariant polynomials of degree n on $\mathfrak{g}$; one has the following theorem.

Theorem 1.5. *The d and the δ cohomologies of $\mathfrak{B}$ are given by*

$$\begin{cases} H^{k,l}(\mathfrak{B}, d) &= 0 \quad \textit{if } l \neq 0 \textit{ or if } k \textit{ is odd}, \\ H^{2n,0}(\mathfrak{B}, d) &= \{P(F) \mid P \in \mathcal{I}_S^n(\mathfrak{g})\} \simeq \mathcal{I}_S^n(\mathfrak{g}) \end{cases}$$

and

$$\begin{cases} H^{k,l}(\mathfrak{B}, \delta) &= 0 \quad \textit{if } k \neq 0 \textit{ or if } l \textit{ is odd}, \\ H^{0,2n}(\mathfrak{B}, \delta) &= \{P(\varphi) \mid P \in \mathcal{I}_S^n(\mathfrak{g})\} \simeq \mathcal{I}_S^n(\mathfrak{g}). \end{cases}$$

Proof. By Lemma 1.4, d_1' (resp. δ_1') gives a homotopy for d (resp. δ) for terms which contain $(\psi, \varphi, B, \beta)$ (resp. (F, ψ, B, β)), so we are left with $P(F)$ (resp. $P(\varphi)$) with $P \in \mathcal{I}_S^n(\mathfrak{g})$.

These cocycles are classically cohomologically independent (in fact they describe the basic cohomologies of the Weil algebras $W_d(\mathfrak{g})$ and $W_\delta(\mathfrak{g})$). □

Remark 1.6. One has with obvious notations, for $P \in \mathcal{I}_S^n(\mathfrak{g})$

$$\begin{aligned} dP(\varphi, \ldots, \varphi) &= nP(\varphi, \ldots, \varphi, d\varphi + [A, \varphi]) \\ &= nP(\varphi, \ldots, \varphi, \beta) = \delta(-nP(\varphi, \ldots, \varphi, \psi)) \end{aligned}$$

and

$$\delta P(F, \ldots, F) = nP(F, \ldots, F, B) = d(-nP(F, \ldots, F, \psi))$$

which means that

$$d(H(\mathfrak{B}, \delta)) = 0 \quad \text{and} \quad \delta(H(\mathfrak{B}, d)) = 0$$

or equivalently,

$$H(H(\mathfrak{B}, \delta), d) = H(\mathfrak{B}, \delta) = \{P(\varphi) \mid P \in \mathcal{I}_S(\mathfrak{g})\} \simeq \mathcal{I}_S(\mathfrak{g})$$

and

$$H(H(\mathfrak{B}, d), \delta) = H(\mathfrak{B}, d) = \{P(F) \mid P \in \mathcal{I}_S(\mathfrak{g})\} \simeq \mathcal{I}_S(\mathfrak{g}).$$

Finally, let us notice that one has:

$$(d + \delta)F + [A + \alpha, F] = B,$$

$$(d + \delta)(F + \psi + \varphi) + [A + \alpha, F + \psi + \varphi] = 0,$$

$$(d + \delta)\varphi + [A + \alpha, \varphi] = \beta,$$

$$(d + \delta)B + [A + \alpha, B] = -[F, \psi] + [\varphi, F],$$

$$(d + \delta)\beta + [A + \alpha, \beta] = -[\varphi, \psi] + [F, \varphi].$$

Therefore one can apply the same method as in the proof of Theorem 1.5, by changing $(F, \psi, \varphi, B, \beta)$ into $(F, F + \psi + \varphi, \varphi, B, \beta)$, to prove the following theorem.

Theorem 1.7. *The $d + \delta$ cohomology of $\mathfrak{B}$ is given by*

$$\begin{cases} H^m(\mathfrak{B}, d + \delta) = 0 & \textit{if } m \textit{ is odd} \\ H^{2n}(\mathfrak{B}, d + \delta) = \{P(F + \psi + \varphi) \mid P \in \mathcal{I}_S^n(\mathfrak{g})\} \simeq \mathcal{I}_S^n(\mathfrak{g}). \end{cases}$$

The $d + \delta$ cohomology is of course only graded by the total degree. Notice that $A + \alpha$ and $F + \psi + \varphi$ generate a third Weil algebra $W_{d+\delta}(\mathfrak{g})$ with differential $d + \delta$.

1.3. The δ cohomology modulo d of $\mathfrak{B}(\mathfrak{g})$

The computation of the d and the δ cohomologies of $\mathfrak{B}$ was a necessary step for the computation of the δ cohomology modulo d of $\mathfrak{B}$ which, as will become clear later on, contains all relevant cohomological information on $\mathfrak{B}$.

Theorem 1.8. *One has $H^{k,l}(\mathfrak{B}, \delta \bmod (d)) = H^{k,l}(\mathfrak{B}, d \bmod (\delta)) = 0$ if $k+l$ is odd and $H^{k,l}(\mathfrak{B}, \delta \bmod (d)) \simeq H^{k,l}(\mathfrak{B}, d \bmod (\delta)) \simeq \mathcal{I}_S^n(\mathfrak{g})$ if $k+l = 2n$. More precisely, on $\mathfrak{B}$, a complete system of cohomologically independent δ cocycles modulo d in bidegree $(m, 2n-m)$ which is also a complete system of cohomologically independent d cocycles modulo δ is given by*

$$\{\textit{term of bidegree } (m, 2n-m) \textit{ in } P(F+\psi+\varphi) \mid P \in \mathcal{I}_S^n(\mathfrak{g})\}.$$

Proof. Let $Q^{k,l}$ be an element of $Z^{k,l}(\mathfrak{B}, \delta \bmod (d))$ and let us denote by $[Q^{k,l}]$ its class in $H^{k,l}(\mathfrak{B}, \delta \bmod (d))$. By definition there is a $Q^{k-1,l+1} \in \mathfrak{B}^{k-1,l+1}$ such that $\delta Q^{k,l} + dQ^{k-1,l+1} = 0$. By applying d one gets $\delta d Q^{k,l} = -d\delta Q^{k,l} = 0$ which implies, since $H^{k+1,l}(\mathfrak{B}, \delta) = 0$, that there is a $Q^{k+1,l-1} \in \mathfrak{B}^{k+1,l-1}$ such that $\delta Q^{k+1,l-1} + dQ^{k,l} = 0$. If $[Q^{k,l}] = 0$, i.e. if $Q^{k,l} = \delta L^{k,l-1} + dL^{k-1,l}$ with $L^{k,l-1}, L^{k-1,l} \in \mathfrak{B}$, then one has $\delta(Q^{k+1,l-1} - dL^{k,l-1}) = 0$ and therefore, since $H^{k+1,l-1}(\mathfrak{B}, \delta) = 0$, there is a $L^{k+1,l-2} \in \mathfrak{B}$ such that $Q^{k+1,l-1} = \delta L^{k+1,l-2} + dL^{k,l-1}$. Thus $[Q^{k,l}] = 0$ implies $[Q^{k+1,l-1}] = 0$ and therefore there is a well-defined linear mapping $\partial' : H^{k,l}(\mathfrak{B}, \delta \bmod (d)) \to H^{k+1,l-1}(\mathfrak{B}, \delta \bmod (d))$ defined by $\partial'[Q^{k,l}] = [Q^{k+1,l-1}]$.

∂' is injective for $(k,l) \neq (2n, 0)$. Indeed, assume $\partial'[Q^{k,l}] = 0$ or, equivalently $Q^{k+1,l-1} = \delta L^{k+1,l-2} + dL^{k,l-1}$ with $L^{k+1,l-2}, L^{k,l-1} \in \mathfrak{B}$; then $dQ^{k,l} = d\delta L^{k,l-1}$ $(= -\delta d L^{k,l-1})$ which implies $[Q^{k,l}] = 0$ if $(k,l) \neq (2n, 0)$ since then $H^{k,l}(\mathfrak{B}, d) = 0$.

∂' is surjective. Indeed let $Q^{k+1,l-1}$ be a δ cocycle modulo d of $\mathfrak{B}$, i.e. there is a $Q^{k,l} \in \mathfrak{B}^{k,l}$ such that $\delta Q^{k+1,l-1} + dQ^{k,l} = 0$; then $d\delta Q^{k,l} = 0$ which implies that $Q^{k,l}$ is also a δ cocycle modulo d of $\mathfrak{B}$ since $H^{k,l+1}(\mathfrak{B}, d) = 0$ and therefore one has $[Q^{k+1,l-1}] = \partial' J[Q^{k,l}]$.

We proved that $\partial' : H^{k,l}(\mathfrak{B}, \delta \bmod (d)) \to H^{k+1,l-1}(\mathfrak{B}, \delta \bmod (d))$ *is an isomorphism for* $(k,l) \neq (2n, 0)$, which implies that

$$\begin{aligned} \mathcal{I}_S^n(\mathfrak{g}) &= H^{0,2n}(\mathfrak{B}, \delta) \overset{\partial'}{\simeq} H^{1,2n-1}(\mathfrak{B}, \delta \bmod (d)) \\ &\overset{\partial'}{\simeq} \cdots \overset{\partial'}{\simeq} H^{2n-1,1}(\mathfrak{B}, \delta \bmod (d)) \overset{\partial'}{\simeq} H^{2n,0}(\mathfrak{B}, d), \end{aligned}$$

where, for the last isomorphism, we use $H^{2n,0}(\mathfrak{B}, \delta \bmod (d)) \simeq H^{2n,0}(\mathfrak{B}, d)$ which is implied by Remark 1.6.

One can use a similar argument for $H(\mathfrak{B}, d \bmod (\delta))$.

Finally, if $P \in \mathcal{I}_S^n(\mathfrak{g})$, then $(d+\delta)P(F+\psi+\varphi) = 0$ follows from $(d+\delta)(F+\psi+\varphi) = [F+\psi+\varphi, A+\alpha]$ and therefore if one defines $Q^{k,l}$ = term of bidegree (k,l) in the expansion of $P(F+\psi+\varphi)$ for $k+l = 2n$, one has $\delta Q^{k,l} + dQ^{k-1,l+1} = 0$, i.e. $[Q^{k,l}] = \partial'[Q^{k-1,l+1}]$. □

2. Characteristic homomorphism—generalization of the Weil homomorphism

2.1. General framework

We adopt the general framework of [1] (see also in [2]). Let M be a smooth finite dimensional manifold and $P \to M$ be a smooth principal bundle over M with structure group G such that $Lie(G) = \mathfrak{g}$. The gauge group (vertical automorphisms) $\mathcal{G}$ of P acts on the affine space $\mathcal{C}$ of connections on P. Let $\mathcal{P}(\subset \mathcal{C})$ be a $\mathcal{G}$-invariant smooth manifold of connections on P. On the data $(\mathcal{P}, \mathcal{G})$ we assume the following regularity condition: The quotient $\mathcal{M} = \mathcal{P}/\mathcal{G}$, i.e. the set of orbits, is a smooth manifold in such a way that $\mathcal{P} \to \mathcal{M}$ is a smooth principal $\mathcal{G}$-bundle.

For instance one can take $\mathcal{P}$ to be the space of irreducible connections on P and $\mathcal{G}$ to be the group of all vertical automorphisms of P, or one can take $\mathcal{P}$ to be the space of all connections on P and $\mathcal{G}$ to be the group of pointed gauge transformations, i.e. the group of vertical automorphisms leaving invariant one point, and therefore one fibre, of P (all this with appropriate smooth structure). Another classical example is, for M = a compact connected oriented 4-dimensional riemannian manifold and G compact (e.g. $G = SU(2)$), to take $\mathcal{P}$ to be the space of irreducible self-dual connections; in this case $\mathcal{M}$ is a finite dimensional manifold. In any case $\mathcal{M}$ plays the role of a moduli space of connections on P.

The algebra $\Omega(P \times \mathcal{P})$ of differential forms on $P \times \mathcal{P}$ is a bigraded differential algebra with differential $d+\delta$, where d is the exterior differential of P and δ is the exterior differential of $\mathcal{P}$; d is of bidegree $(1, 0)$ and δ is of bidegree $(0, 1)$. Tangent vectors to $P \times \mathcal{P}$ split in two parts, $X = X^{1,0} + X^{0,1}$, where $X^{1,0}$ is tangent to P and $X^{0,1}$ is tangent to $\mathcal{P}$.

There is *a canonical Lie(G)-valued one form* $\underline{A}$ on $P \times \mathcal{P}$ of bidegree $(1,0)$, i.e. $\underline{A} \in \mathfrak{g} \otimes \Omega^{1,0}(P \times \mathcal{P})$, defined by $\forall(\xi, a) \in P \times \mathcal{P}$, $\underline{A}(\xi, a)$ is the connection form at $\xi \in P$ of the connection $a \in \mathcal{P}$, (a is a connection on P).

The algebra $\Omega(M \times \mathcal{M})$ of differential forms on $M \times \mathcal{M}$ is canonically a bigraded differential subalgebra of $\Omega(P \times \mathcal{P})$ which we shall identify as the base of two operations on $\Omega(P \times \mathcal{P})$.

The structure group G of P acts unambiguously on $P \times \mathcal{P}$ via the right action on P. There is a corresponding operation, in the sense of Cartan, of $\mathfrak{g} = Lie(G)$ in $\Omega(P \times \mathcal{P})$: For $X \in \mathfrak{g}$, i_X denotes the inner antiderivation of $\Omega(P \times \mathcal{P})$ by the vertical vector field of type $(1,0)$ corresponding to X (infinitesimal action of G). One has $i_X \underline{A} = X$ and $i_X\delta + \delta i_X = 0$ so the corresponding Lie derivation L_X is given by $L_X = i_X d + d i_X$ and $L_X \underline{A} = [\underline{A}, X]$.

There are several possible actions of $\mathcal{G}$ on $P \times \mathcal{P}$. One is the simultaneous action on P and on $\mathcal{P}$, another one is the action on $\mathcal{P}$ only. For the sequel (i.e. the characterization of $\Omega(M \times \mathcal{M})$ in $\Omega(P \times \mathcal{P})$), one can use the operation corresponding to any of these two actions. The simultaneous action might seem more natural since, after all, $\mathcal{G}$ is defined by its action on P (vertical automorphisms). Nevertheless we choose the second one because the corresponding operation seems to us easier to describe; in this case, the infinitesimal action of $\mathcal{G}$ corresponds to vertical vector fields of type (0,1). The Lie algebra $Lie(\mathcal{G})$ identifies with $Lie(G)$-valued equivariant functions on P, $\mathfrak{X} : P \to \mathfrak{g} = Lie(G)$. For $\mathfrak{X} \in Lie(\mathcal{G})$, $\mathfrak{i}_{\mathfrak{X}}$ denotes the inner antiderivation of $\Omega(P \times \mathcal{P})$ by the vertical vector field of type

(0, 1) corresponding to $\mathfrak{X}$. One has $\mathrm{i}_{\mathfrak{X}} d + d\mathrm{i}_{\mathfrak{X}} = 0$ so the corresponding Lie derivative $\mathfrak{L}_{\mathfrak{X}}$ is given by $\mathfrak{L}_{\mathfrak{X}} = \mathrm{i}_{\mathfrak{X}}\delta + \delta\mathrm{i}_{\mathfrak{X}}$. Furthermore one has $\mathrm{i}_{\mathfrak{X}}(\underline{A}) = 0$ and $\mathfrak{L}_{\mathfrak{X}}(\underline{A}) = d\mathfrak{X} + [\underline{A}, \mathfrak{X}]$ with obvious notations (reminding that $\mathfrak{X}$ is a $Lie(G)$-valued function). The last equality reflecting the infinitesimal action of the group of gauge transformations on the space of connections which can be summarized by $\underline{A} \mapsto g^{-1}dg + g^{-1}\underline{A}g$ with our notations; this is a right action on $\mathcal{P}$.

$\Omega(M \times \mathcal{M})$ is the set of basic elements of $\Omega(P \times \mathcal{P})$ for these operations, i.e. $\cap\{\ker(i_X) \cap \ker(L_{X'}) \cap \ker(\mathrm{i}_{\mathfrak{X}}) \cap \ker(\mathfrak{L}'_{\mathfrak{X}})\}$, $X, X' \in Lie(G), \mathfrak{X}, \mathfrak{X}' \in Lie(\mathcal{G})$.

2.2. Characteristic homomorphism

It is well known that there exist connections on principal $\mathcal{G}$-bundles $\mathcal{P} \to \mathcal{M}$ for G compact (for instance there are such connections associated to choices of riemannian structures on M). In any case let $\mathcal{A}$ be a connection on $\mathcal{P}$. The corresponding connection form is a $Lie(\mathcal{G})$-valued one-form on $\mathcal{P}$. Remembering that $Lie(\mathcal{G})$ consists of $Lie(G)$-valued functions on P, it follows, by evaluation on P, that this connection form defines a $Lie(G)$-valued one-form $\underline{\alpha}$ on $P \times \mathcal{P}$ of bidegree (0, 1), i.e. $\underline{\alpha} \in \mathfrak{g} \otimes \Omega^{0,1}(P \times \mathcal{P})$. One has $\mathrm{i}_{\mathfrak{X}}(\underline{\alpha}) = \mathfrak{X}$ and $\mathfrak{L}_{\mathfrak{X}}(\underline{\alpha}) = [\underline{\alpha}, \mathfrak{X}]$ for $\mathfrak{X} \in Lie(\mathcal{G})$ and $i_X(\underline{\alpha}) = 0$ and $L_X(\underline{\alpha}) = [\underline{\alpha}, X]$ for $X \in Lie(G)$.

By Lemma 1.2, there is a unique homomorphism of bigraded differential algebras $h : \mathfrak{A} \to \Omega(P \times \mathcal{P})$ such that $h(A) = \underline{A}$ and $h(\alpha) = \underline{\alpha}$. This homomorphism satisfies $i_X \circ h = h \circ i_X$, $\forall X \in \mathfrak{g}$. The following lemma is the justification of the same base of $\mathfrak{A}$ for the subalgebra $\mathfrak{B}$.

Lemma 2.1. *One has* $h(\mathfrak{A}) \cap \Omega(M \times \mathcal{M}) = h(\mathfrak{B})$.

Proof. One has $i_X(h(A)) = i_X(\underline{A}) = X$ for $X \in Lie(G)$ and $\mathrm{i}_{\mathfrak{X}} h(\alpha) = \mathrm{i}_{\mathfrak{X}}(\underline{\alpha}) = \mathfrak{X}$, $\mathrm{i}_{\mathfrak{X}} h(\xi) = \mathrm{i}_{\mathfrak{X}}(\delta\underline{A}) = \mathfrak{L}_{\mathfrak{X}}(\underline{A}) = d\mathfrak{X} +, [\underline{A}, \mathfrak{X}]$, for $\mathfrak{X} \in Lie(\mathcal{G})$. Therefore, by horizontality, $h(\mathfrak{A}) \cap \Omega(M \times \mathcal{M})$ cannot contain expressions depending on $h(A)$, $h(\alpha)$ and $h(\xi)$; thus elements of $h(\mathfrak{A}) \cap \Omega(M \times \mathcal{M})$ are of the form $h(\mathcal{I}(F, \psi, \varphi, B, \beta))$. On the other hand, if $Z = h(F), h(\psi), h(\varphi), h(B)$ or $h(\beta)$ one has $L_X Z = [Z, X]$ for $X \in Lie(G)$ and $\mathfrak{L}_{\mathfrak{X}} Z = [Z, \mathfrak{X}]$ for $\mathfrak{X} \in Lie(\mathcal{G})$, therefore invariance of $h(\mathcal{I}(F, \psi, \varphi, B, \beta))$ is equivalent to total $ad(\mathfrak{g})$-invariance of $\mathcal{I}$, i.e. $h(\mathfrak{A}) \cap \Omega(M \times \mathcal{M}) \subseteq h(\mathfrak{B})$. In order to achieve the proof of the lemma, (i.e. to show the equality) it remains to show that $\mathrm{i}_{\mathfrak{X}} h(\mathfrak{B}) = 0$. One has (with $\underline{F} = h(F)$, etc.):

$$\mathrm{i}_{\mathfrak{X}}(\underline{F}) = \mathrm{i}_{\mathfrak{X}}(d\underline{A} + \tfrac{1}{2}[\underline{A}, \underline{A}]) = -d\mathrm{i}_{\mathfrak{X}}(\underline{A}) + [\mathrm{i}_{\mathfrak{X}}(\underline{A}), \underline{A}] = 0,$$

$$\begin{aligned}\mathrm{i}_{\mathfrak{X}}(\underline{\psi}) &= \mathrm{i}_{\mathfrak{X}}(\delta\underline{A} + d\underline{\alpha} + [\underline{A}, \underline{\alpha}]) = \mathfrak{L}_{\mathfrak{X}}\underline{A} - d\mathrm{i}_{\mathfrak{X}}(\underline{\alpha}) + [\mathrm{i}_{\mathfrak{X}}(\underline{\alpha}, \underline{A})] \\ &= d\mathfrak{X} + [\underline{A}, \mathfrak{X}] - d\mathfrak{X} + [\mathfrak{X}, \underline{A}] = 0,\end{aligned}$$

$$\begin{aligned}\mathrm{i}_{\mathfrak{X}}(\underline{\varphi}) &= \mathrm{i}_{\mathfrak{X}}(\delta\underline{\alpha} + \tfrac{1}{2}[\underline{\alpha}, \underline{\alpha}]) = \mathfrak{L}_{\mathfrak{X}}\underline{\alpha} + [\mathrm{i}_{\mathfrak{X}}(\underline{\alpha}), \underline{\alpha}] \\ &= [\underline{\alpha}, \mathfrak{X}] + [\mathfrak{X}, \underline{\alpha}] = 0,\end{aligned}$$

$$\begin{aligned}\mathrm{i}_{\mathfrak{X}}(\underline{B}) &= \mathrm{i}_{\mathfrak{X}}(\delta\underline{F} + [\underline{\alpha}, \underline{F}]) = \mathfrak{L}_{\mathfrak{X}}\underline{F} + [\mathrm{i}_{\mathfrak{X}}(\underline{\alpha}), \underline{F}] \\ &= [\underline{F}, \mathfrak{X}] + [\mathfrak{X}, \underline{F}] = 0,\end{aligned}$$

$$\mathrm{i}_{\mathfrak{X}}(\underline{\beta}) = \mathrm{i}_{\mathfrak{X}}(d\underline{\varphi} + [\underline{A}, \underline{\varphi}]) = -d\mathrm{i}_{\mathfrak{X}}(\underline{\varphi}) - [\underline{A}, \mathrm{i}_{\mathfrak{X}}(\underline{\varphi})] = 0,$$

which implies in particular $\mathrm{i}_{\mathfrak{X}} h(\mathfrak{B}) = 0$. □

It follows from this lemma that h induces homomorphisms of the various cohomologies of $\mathfrak{B}$ in the corresponding cohomologies of $\Omega(M \times \mathcal{M})$, in particular, h induces an homomorphism h^c of $H(\mathfrak{B}, \delta \bmod (d))$ in the δ cohomology modulo d of $\Omega(M \times \mathcal{M})$. Of course h depends on $\mathcal{A}$, however one has the following theorem.

Theorem 2.2. *The homomorphism h^c is independent of $\mathcal{A}$.*

Proof. Let $\mathcal{A}_t$, $t \in [0, 1]$, be a smooth family of connections on $\mathcal{P} \to \mathcal{M}$ and let h_t be the corresponding family of homomorphisms of bigraded differential algebras of $\mathfrak{A}$ in $\Omega(P \times \mathcal{P})$. One has for $P \in \mathcal{I}_S^n(\mathfrak{g})$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t} h_t(P(F + \psi + \varphi, \ldots, F + \psi + \varphi)) \\ &\quad = (d + \delta)\left(nP\left(\frac{\mathrm{d}}{\mathrm{d}t} h_t(\alpha), h_t(F + \psi + \varphi), \ldots, h_t(F + \psi + \varphi)\right)\right).\end{aligned}$$

On the other hand $nP(\mathrm{d}h_t(\alpha)/\mathrm{d}t, h_t(F + \psi + \varphi), \ldots, h_t(F + \psi + \varphi))$ is basic for the operations of $Lie(G) = \mathfrak{g}$ and of $Lie(\mathcal{G})$ and therefore it is an element of $\Omega(M \times \mathcal{M})$ so, by integration one has

$$h_1(P(F + \psi + \varphi)) - h_0(P(F + \psi + \varphi)) \in (d + \delta)\Omega(M \times \mathcal{M})$$

which means, in view of Theorem 1.8 that $h_1^c - h_0^c = 0$. □

The homomorphism h^c will be called *the characteristic homomorphism.*

Remark 2.3.

(a) The same proof shows that the homomorphism $\tilde{h}^c$ of $H(\mathfrak{B}, d \bmod (\delta))$ in the d cohomology modulo δ of $\Omega(M \times \mathcal{M})$ is independent of the choice of $\mathcal{A}$. This also shows that the homomorphism $h_{d+\delta}^c$ of the $d + \delta$ cohomology of $\mathfrak{B}$ in the total de Rham cohomology of $M \times \mathcal{M}$ does not depend on $\mathcal{A}$. Of course h^c, $\tilde{h}^c$ and $h_{d+\delta}^c$ are essentially the same thing in view of Theorems 1.7 and 1.8.

(b) Similarly, $h_\delta^c : H(\mathfrak{B}, \delta) \to H(\Omega(M \times \mathcal{M}), \delta)$ is independent of $\mathcal{A}$ because

$$\frac{\mathrm{d}}{\mathrm{d}t} h_t(P(\varphi)) = \delta\left(nP\left(\frac{\mathrm{d}}{\mathrm{d}t} h_t(\alpha), h_t(\varphi), \ldots, h_t(\varphi)\right)\right)$$

and

$$nP\left(\frac{\mathrm{d}}{\mathrm{d}t} h_t(\alpha), h_t(\varphi), \ldots, h_t(\varphi)\right) \in \Omega(M \times \mathcal{M}).$$

(c) Finally, $h_d^c : H(\mathfrak{B}, d) \to H(\Omega(M \times \mathcal{M}), d)$ is independent of $\mathcal{A}$ since $h(P(F))$ is already independent of $\mathcal{A}$. In fact h_d^c is the Weil homomorphism for $P \to M$,

$$h_d^c : H(\mathfrak{B}, d) \simeq \mathcal{I}_S(\mathfrak{g}) \xrightarrow{ch} H(M) = H(\Omega(M), d) \subset H(\Omega(M \times \mathcal{M}), d),$$

where ch is the usual Chern character of $P \to M$.

(d) The Weil homomorphism $ch \simeq h_d^c$ is already included in h^c since $h^c \restriction H^{*,0}(\mathfrak{B}, \delta \bmod (d)) \simeq h_d^c$. In the same way,

$$h^c \restriction H^{0,*}(\mathfrak{B}, \delta \bmod (d)) \simeq h_\delta^c,$$

so the characteristic homomorphims h^c contains all the relevant information and generalizes the Weil homomorphism in an obvious sense.

2.3. *Cartan operations in $\mathfrak{A}(\mathfrak{g})$ and $h(\mathfrak{A}(\mathfrak{g}))$*

Let us denote by $\underline{Z}$ the image by h of the generator $Z = A, F, \alpha, \varphi, \psi, \xi, B, \beta$ of $\mathfrak{A}$ i.e. $\underline{Z} = h(Z)$. Then the Cartan operations of $\mathfrak{g} = Lie(G)$ and of $Lie(\mathcal{G})$ satisfy the following relations: ($X \in \mathfrak{g}, \mathfrak{X} \in Lie(\mathcal{G})$) $i_X(\underline{A}) = X$ and $i_X(\underline{Z}) = 0$ for the other Z, ($\delta X = dX = 0$), $i_X\delta + \delta i_X = 0$ so $L_X = i_X d + d i_X$ and $L_X(\underline{Z}) = [\underline{Z}, X]$ for $Z = A, F, \alpha, \ldots, \beta$; $\mathrm{i}_{\mathfrak{X}}(\underline{\alpha}) = \mathfrak{X}$, $\mathrm{i}_{\mathfrak{X}}(\underline{\xi}) = d\mathfrak{X} + [\underline{A}, \mathfrak{X}]$ and $\mathrm{i}_{\mathfrak{X}}(\underline{Z}) = 0$ for the other Z, $\mathrm{i}_{\mathfrak{X}} d + d\mathrm{i}_{\mathfrak{X}} = 0$, ($\delta\mathfrak{X} = 0$), so $\mathfrak{L}_{\mathfrak{X}} = \mathrm{i}_{\mathfrak{X}} J\delta + \delta \mathrm{i}_{\mathfrak{X}}$ and then $\mathfrak{L}_{\mathfrak{X}}(\underline{A}) = d\mathfrak{X} + [\underline{A}, \mathfrak{X}]$ and $\mathfrak{L}_{\mathfrak{X}}(\underline{Z}) = [\underline{Z}, \mathfrak{X}]$ for $Z = F, \alpha, \ldots, \beta$.

One thus sees that the operation of $\mathfrak{g} = Lie(G)$ corresponds under h to the operation $X \mapsto i_X$ of $\mathfrak{g}$ in $\mathfrak{A}$ defined in Section 1. In contrast, there is a problem to define an analogue in $\mathfrak{A}$ of the operation of $Lie(\mathcal{G})$.The reason is that $d\mathfrak{X}$ does not mean anything in $\mathfrak{A}$. So, in order to define the analogue of this operation in $\mathfrak{A}$ one has to add generators, for instance one may combine $\mathfrak{A}(\mathfrak{g})$ with the Weil–BRS algebra $\mathcal{A}(\mathfrak{g})$ [7] but the result is complicated and the interest of the universal model $(\mathfrak{A}, \mathfrak{B})$ is its simplicity. That is why we refrain to follow this way and defined directly the base $\mathfrak{B}$ of $\mathfrak{A}$ in Section 1.

However, in contrast to the operation $X \mapsto i_X$, the operation $X \mapsto i'_X$ does not correspond to an operation in $\Omega(P \times \mathcal{P})$ under the homomorphism h. Thus the property of $\mathfrak{B}$ to be basic for i' (and not only for i) looks accidental in the context used here but it is an indication that there may be another context in which $(\mathfrak{A}, \mathfrak{B}')$ plays the role of a universal model.

2.4. *Connection with Donaldson invariants*

Let M be an oriented conformal 4-dimensional compact manifold, $P \to M$ be as above with G compact and take for $\mathcal{P}$ the space of self-dual connections on P, i.e. $\mathcal{M}$ is the moduli space of instantons and is finite dimensional. Let ω^{k_i, l_i} be elements of bidegrees (k_i, l_i) in the image of h^c such that $\sum_i l_i = \dim(\mathcal{M})$ and let C_{k_i} be elements of $H_{k_i}(M)$. Then $\langle \omega^{k_i, l_i}, C_{k_i} \rangle$ are in $H^{l_i}(\mathcal{M})$ and $\bigcup_i \langle \omega^{k_i, l_i}, C_{k_i} \rangle$ is in $H^{\dim(\mathcal{M})}(\mathcal{M})$. Therefore by coupling $\bigcup_i \langle \omega^{k_i, l_i}, C_{k_i} \rangle$ with the fundamental class of $\mathcal{M}$, i.e. by integrating over $\mathcal{M}$ one produces numbers which are the Donaldson invariants; these invariants are polynomials on $H_*(M)$

(in fact on $H_2(M)$ if M is simply connected). The non-triviality of these invariants implies that generically the image of the characteristic homomorphism h^c is non-trivial.

3. Conclusions

By construction, the homomorphism h maps $\mathfrak{A}$ onto the algebra of differential forms on $P \times \mathcal{P}$ generated by $\underline{A}$ (the gauge potentials), by $\underline{\alpha}$ (the ghosts) and by their δ and d differentials (i.e. their BRS and exterior differentials). Lemma 2.1 shows that among these differential forms, the one living on $M \times \mathcal{M}$ are exactly the one which are in the image of the basis $\mathfrak{B}$ and Theorem 1.8 states that the δ-cohomology modulo d of $\mathfrak{B}$ is exactly given by the bigraded expansions of the Chern–Weil polynomials. This justifies the claim in the introduction that the (local) relevant basic BRS-cohomology of the bigraded differential algebra generated by the gauge potentials, the ghosts and their BRS and exterior differentials is exhausted by the classes corresponding to the bigraded expansions of the Chern–Weil cocycles. Of course, from the point of view of quantum field theory, one would like to add arbitrary derivatives of the fields and not only the one entering through their differentials in the above sense. In this respect our result is a priori incomplete for field theory, however one may expect that, as for the case of ordinary gauge theory [8], inclusion of such terms does not modify fundamentally the result. As pointed out to us by R. Stora, it is unclear whether, in this context, the locality has to be assumed so also in this respect our result is incomplete from the point of view of topological field theory. In any case we have described here a very general frame to produce invariants which is a direct extension of the Weil algebra approach to characteristic classes and makes contact with topological field theory.

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